Journal of Computational and Applied Mathematics 14 (1986) 111–123
North-Holland

Numerical conformal mapping via the Szegö kernel *

Norberto KERZMAN
University of North Carolina, Department of Mathematics, Chapel Hill, NC 27514, U.S.A.

Manfred R. TRUMMER **
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

Received 10 April 1984
Revised 22 September 1984

Abstract: A new method to compute the Riemann mapping function is numerically implemented and tested on examples. The method expresses the Szegö kernel as the solution of a second-kind integral equation. The equation has its origin in an earlier non-numerical work of Stein and one of the authors [8]. The experimental results show the algorithm to be effective and stable.

Keywords: Conformal mapping, Szegö kernel, integral equations.

1. Introduction

The Riemann mapping function of a smooth, bounded, simply connected domain Ω in the plane can be easily written down in terms of the Szegö kernel (briefly, S) of Ω. The relevant formula, (2.1) below, is classical and quite explicit. The reader can find the definition and basic properties of S in, e.g., [2,5,7,12]. In spite of such a simple relationship, the Szegö kernel has not become a preferred method in numerical conformal mapping. The reason is that S arises from orthonormalization of monomials (in the linear measure of the boundary, i.e., in arc length). The usual Gram–Schmidt procedure called upon to perform this task is considered to be numerically unstable and, hence, unsatisfactory. A similar drawback affects the Bergman kernel, which is a somewhat more familiar relative of S, and which arises when the two-dimensional plane measure of Ω is considered. Indeed, the preferred and more developed conformal mapping methods, throroughly surveyed in [5,7], rely on integral equations and not on the above kernels.

In this paper we shall compute S as the solution of a new, numerically tractable, integral equation of the second kind. Orthonormalization is totally avoided. The equation is (2.8) and (2.9) below. Its explicit, smooth, skew-hermitian kernel vanishes on the diagonal and has a

* Research partially supported by NSF grant MCS-8001887 (U.S.A.)
** This work was performed at the Department of Mathematics, University of North Carolina, Chapel Hill, NC, U.S.A.

geometric interpretation. Actually, the kernel of the equation is the difference between the Cauchy kernel and its adjoint. The parametrization of the boundary of Ω need not be arc length. The theoretical underpinnings of the present work originate in [8]. Our main points are: The reformulation of the basic result of [8] in terms of an integral equation; the numerical implementation of the equation using Nyström's method; and the testing of the foregoing on examples (computation of the Riemann mapping function). Notice also the simple minded but useful formula (2.2) that expresses the mapping on the boundary in terms of its derivative, without recourse to integration.

Organization of this paper: Section 2 contains the statement of the method; Section 3, of theoretical nature and relying on [8], has the derivation of the integral equation plus proofs not already in [8]. The reader willing to believe the results can skip Section 3 without loss of continuity. A brief, informal explanation of the basic idea underlying the integral equation is also included, referring to [8] for technical details; Section 4 has the numerical implementation, and Section 5 has examples and numerical results.

2. Description of the method

We start with an open, bounded, simply connected domain Ω in the complex plane $\mathbb{C}$ and assume that its boundary $\partial\Omega$ is twice continuously differentiable. That is, $\partial\Omega$ admits a counterclockwise parametrization $z(t)$, $0 \leqslant t \leqslant \beta$, $z(0) = z(\beta)$, $\dot{z}(0) = \dot{z}(\beta)$, $\ddot{z}(0) = \ddot{z}(\beta)$, with $\dot{z}(t) = \mathrm{d}z/\mathrm{d}t \neq 0$ for all t. *The parameter t need not be arc length.* The unit tangent to $\partial\Omega$ at $z(t)$ is denoted by $\dot{\gamma}(z) = \dot{z}(t)/|\dot{z}(t)|$.

We want to compute the Riemann mapping function R: $\Omega \to$ Unit Disk subject to the usual normalization $R(a) = 0$, $R'(a) > 0$, where $a \in \Omega$ is an arbitrary point that will remain fixed throughout. It is known a priori that $R'(z)$ has a continuous extension to the closure $\bar{\Omega}$ because $\partial\Omega$ is of class C^2 (this is a classical theorem of Kellog; see, e.g. [11]). To carry out the computation we use, as an intermediate step, the known connection between R and the Szegö kernel $S(z, a)$ of Ω.[1]

Theorem 1. *The Szegö kernel $S(z, a)$ is continuous as a function of z on $\bar{\Omega}$ and*

$$R'(z) = (2\pi/S(a, a))S^2(z, a), \quad z \in \bar{\Omega}. \tag{2.1}$$

Moreover $R'(z)$ yields $R(z)$ (without any integration) by means of

$$R(z) = (1/i)\dot{\gamma}(z)R'(z)/|R'(z)|, \quad z \in \partial\Omega. \tag{2.2}$$

Notice that (2.1) and (2.2) give the boundary correspondence as well as its derivative in terms of $S(z, a)$. Hence the problem is to compute $S(z, a)$. To do so we resort to the following kernel that measures "how much the Cauchy kernel deviates from being hermitian". This kernel, introduced in [8], is the essence of our method. It is also the kernel of the integral equation (2.8) that yields S.

[1] Ω means the closure of Ω, while $\bar{z}$ stands for the conjugate of z. In other similar cases, the meaning of the bar will be clear from the context. The reference for (2.1) is [2]. See also [8] and Section 3 below, where (2.2) is proved; other useful references are [5,7,12].

Definition of the kernel $A(w, z)$. For $w \in \bar{\Omega}$, $z \in \partial\Omega$, $w \neq z$, set

$$H(w, z) \overset{\text{Def}}{=} \frac{1}{2\pi i} \frac{\dot{\gamma}(z)}{z - w} \tag{2.3}$$

and

$$A(w, z) \overset{\text{Def}}{=} \begin{cases} H(z, w) - \overline{H(w, z)}, & w \in \partial\Omega,\ z \in \partial\Omega,\ w \neq z \\ 0, & w = z \in \partial\Omega, \end{cases} \tag{2.4}$$

so that A: $\partial\Omega \times \partial\Omega \to \mathbb{C}$.

Observe that $H(w, z)$ is the coefficient appearing in front of the arc element $d\sigma_z$ in the Cauchy kernel expression

$$\frac{1}{2\pi i} \frac{dz}{z - w} \tag{2.5}$$

because $dz = \dot{\gamma}(z) d\sigma_z$.

The key properties of A, proved in [8] (with some complements and remarks in Section 3 below) are

Theorem 2. (a) *$A(w, z)$ is a continuous function on $\partial\Omega \times \partial\Omega$. If $\partial\Omega$ is of class C^k then A is C^{k-2} on $\partial\Omega \times \partial\Omega$, $k \geq 2$.*

(b) $A(w, z) = -\overline{A}(z, w)$, $w \in \partial\Omega$, $z \in \partial\Omega$.

(c) *$A(w, z)$ has the following geometric interpretation*:

$$A(w, z) = \frac{1}{2\pi i} \frac{1}{z - w} [\tilde{\gamma}(w) - \dot{\gamma}(z)], \quad w \neq z, \tag{2.6}$$

where $\tilde{\gamma}(w)$ stands for the vector obtained by reflecting the unit tangent $\dot{\gamma}$-(w) in the chord that joins z to w. (see Fig. 1).

Observe that a cancellation of singularities has taken place: $H(w, z)$ and $\overline{H}(z, w)$ both blow up at $z = w$ but their differences is continuous. On the other hand $A(w, z)$ is identically zero, if Ω is a disk because in this case chords meet the circle at equal angles on both end points. But the disk is the only region for which this can happen because this equal angle property characterizes

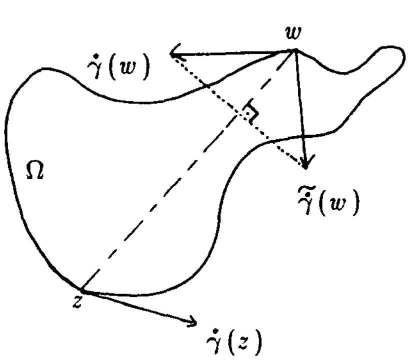

Fig. 1. Geometric interpretation of the kernel A.

the circle among all possible $\partial\Omega$. See [13]. (The geometric interpretation of A, while not needed for the numerical computations, may be enlightening.)

Finally, our main result is:

The Szegö kernel is the solution of an integral equation with kernel A.

In fact, let $g(z)$ be the (known) function defined on $\partial\Omega$ by

$$g(z) \overset{\text{Def}}{=} \frac{1}{2\pi i}\overline{[\dot{\gamma}(z)/(a-z)]}, \quad z \in \partial\Omega, \tag{2.7}$$

and let $f(z) = S(z, a)$, $z \in \partial\Omega$ (which we are seeking to compute). Then

Theorem 3. *The function f is the unique solution (in the class of continuous functions) of the integral equation*

$$f(z) + \int_{w \in \partial\Omega} A(z, w) f(w)\, \mathrm{d}\sigma_w = g(z), \quad z \in \partial\Omega, \tag{2.8}$$

where $\mathrm{d}\sigma$ *stands for arc length on* $\partial\Omega$.

The proof is in Section 3 below.

Finally, reverting to the original parametrization $z(t)$ of $\partial\Omega$, $0 \leqslant t \leqslant \beta$, integral equation (2.8) becomes

$$\varphi(t) + \int_0^\beta k(t, s)\varphi(s)\mathrm{d}s = \psi(t), \quad 0 \leqslant t \leqslant \beta, \tag{2.9}$$

where we have introduced

$$\varphi(t) = |\dot{z}(t)|^{1/2} f(z(t)), \tag{2.10}$$

$$\psi(t) = |\dot{z}(t)|^{1/2} g(z(t)), \tag{2.11}$$

$$k(t, s) = |\dot{z}(t)|^{1/2} |\dot{z}(s)|^{1/2} A(z(t), z(s)) \tag{2.12}$$

for $0 \leqslant t,\ s \leqslant \beta$. Under this reparametrization, the skew-hermitian property of the kernel is preserved, i.e., $k(s, t) = -\bar{k}(t, s)$.

Remark 1. Relaxing the smoothness of $\partial\Omega$ from C^∞ (which was the stated hypothesis in [8]) to C^2 causes no difficulties, as explained in Section 3.5. But the present paper does not deal with corners.

Remark 2. The Szegö kernel of a *multiply connected* smooth bounded domain can be obtained via integral equation (2.8) without modification. No simple connectivity hypothesis is involved in the proofs of Theorems 2 and 3. How to relate $S(w, z)$ to conformal mapping is a different matter.

Remark 3. Notice that the kernel A we use has no singularity whatsoever and that, besides its Hilbert space meaning, it has the geometric significance exhibited in Fig. 1. For other relationships between the Szegö kernel and integral equations, see [2,9], and their references. (Note that some kernels in the literature are called geometric even though they do not have a pictorial interpretation connected with the shape of the boundary of Ω.)

3. Theoretical background

3.1. The Szegö kernel

The boundary $\partial\Omega$ carries the Hilbert space $L^2 = L^2(\partial\Omega, \mathrm{d}\sigma)$ of complex-valued square integrable functions. The measure $\mathrm{d}\sigma$ is arc length. L^2 contains an interesting closed subspace $\mathscr{H}^2$, called the Hardy space. The elements of $\mathscr{H}^2$ are those $u \in L^2$ for which there is some (necessarily unique) holomorphic function in Ω that assumes u as its boundary values in the L^2 sense. The *orthogonal projector* $\boldsymbol{S}$: $L^2 \to \mathscr{H}^2$ leads to the Szegö kernel $S(w, z)$ as follows: If $u \in L^2$ then

$$\mu(w) \stackrel{\text{Def}}{=} \int_{z \in \partial\Omega} S(w, z) u(z)\, \mathrm{d}\sigma_z, \quad w \in \Omega \tag{3.1}$$

is holomorphic in Ω and assumes $\boldsymbol{S}u$ as its boundary values in the L^2 sense. The notation $\boldsymbol{S}$ for the projector is motivated by its connection (3.1) with $S(w, z)$.

The Szegö kernel $S(w, z)$ is defined and continuous on the set $B \stackrel{\text{Def}}{=} (\bar{\Omega} \times \bar{\Omega}) = \{(z, z);\ z \in \partial\Omega\}$, and is holomorphic in its first variable $w \in \Omega$ and conjugate-holomorphic in the second. Since $\boldsymbol{S}$ is an *orthogonal*projector, it is *self-adjoint*. This fact is reflected in the hermitian property

$$S(w, z) = \bar{S}(z, w), \quad (w, z) \in B. \tag{3.2}$$

3.2. The Cauchy kernel

Starting with $u \in L^2(\partial\Omega)$, set now

$$\nu(w) = \frac{1}{2\pi \mathrm{i}} \int_{z \in \partial\Omega} \frac{u(z)}{z - w} \mathrm{d}z, \quad w \in \Omega. \tag{3.3}$$

The function ν is holomorphic in Ω. The theory of the Hilbert transform shows that a new function $\boldsymbol{H}u \in L^2(\partial\Omega)$ appears such that ν assumes $\boldsymbol{H}u$ as its boundary values in the L^2 sense. This yields a bounded linear operator $\boldsymbol{H}$: $L^2 \to \mathscr{H}^2$, $u \to \boldsymbol{H}u$, which is an *oblique* projector, i.e., $\boldsymbol{H} \cdot \boldsymbol{H} = \boldsymbol{H}$ (by the reproducing property of the Cauchy kernel); but $\boldsymbol{H}$ is not self adjoint in general, i.e. $\boldsymbol{H}^* \neq \boldsymbol{H}$. Using (2.3) and (2.5) we can rewrite (3.3) as

$$\nu(w) = \int_{z \in \partial\Omega} H(w, z) u(z) \mathrm{d}\sigma_z, \quad w \in \Omega. \tag{3.4}$$

Hence, the Cauchy kernel H is related to the *oblique* projector $\boldsymbol{H}$ in the same way that the Szegö kernel S is related to the *orthogonal* projector $\boldsymbol{S}$. The idea is now to study the relationship between S and H via the corresponding one between $\boldsymbol{S}$ and $\boldsymbol{H}$.

3.3. Deviation from self adjointness

Let $A(w, z)$ be the kernel in (2.4). Then, $\boldsymbol{H}^* - \boldsymbol{H}$ admits the following representation for any $u \in L^2(\partial\Omega)$,

$$(\boldsymbol{H}^* - \boldsymbol{H}) u(w) = \int_{z \in \partial\Omega} A(w, z) u(z) \mathrm{d}\sigma_z, \quad w \in \partial\Omega. \tag{3.5}$$

This is *not* immediate because (3.4) holds for $w \in \Omega$ and not for $w \in \partial\Omega$. The proof in [8] makes use of the Plemelj formula. Notice in passing that as a consequence of (3.5), the *Szegö kernel* $S(w, z)$ *and the Cauchy kernel* $H(w, z)$ *are the same if and only if* Ω *is a disk*. This is so because $S \equiv H \leftrightarrow \boldsymbol{S} = \boldsymbol{H} \leftrightarrow \boldsymbol{H}^* = \boldsymbol{H} \leftrightarrow A \equiv 0 \leftrightarrow \Omega$ is a disk, in view of the remark below Theorem 2.

3.4. *Derivation of the integral equation for the Szegö kernel*

We are now going to prove Theorem 3. The starting point, as in [8], is the pair of relationships between operators acting on $L^2(\partial\Omega)$

$$\boldsymbol{HS} = \boldsymbol{S}, \tag{3.6}$$

$$\boldsymbol{SH} = \boldsymbol{H}. \tag{3.7}$$

These hold because both projectors, $\boldsymbol{H}$ and $\boldsymbol{S}$, project onto the same subspace. Taking adjoints and using $\boldsymbol{S}^* = \boldsymbol{S}$ we obtain

$$\boldsymbol{SH}^* = \boldsymbol{S} \tag{3.8}$$

$$\boldsymbol{H}^*\boldsymbol{S} = \boldsymbol{H}^*. \tag{3.9}$$

If we define $\boldsymbol{A} = \boldsymbol{H}^* - \boldsymbol{H}$, (3.7) and (3.8) yield

$$\boldsymbol{H} = \boldsymbol{S} - \boldsymbol{SA}. \tag{3.10}$$

Consider the fixed point $a \in \Omega$ of Section 2 and let us focus our attention on an arbitrary point $z \in \partial\Omega$. Let $\delta = \delta_z$ be the point mass at z with respect to the boundary arc length $\mathrm{d}\sigma$. Let $\{\varphi_j\}$ be a sequence of continuous function $\varphi_j(\xi)$ defined on $\partial\Omega$ and approximating δ_z. For each j, (3.10) yields

$$\boldsymbol{H}\varphi_j = \boldsymbol{S}\varphi_j - \boldsymbol{SA}\varphi_j \tag{3.11}$$

as functions in $\mathscr{H}^2$. But each $v \in \mathscr{H}^2$ is the boundary value in the L^2 sense of a *unique* holomorphic function on Ω which we shall call *the* 'extension' of v and shall denote also by v. What the extensions of the three terms in (3.11) are is not difficult to find out: they are given by (3.1) and (3.3) in view of what is said below those formulas. Evaluation of the extensions at a yields

$$(\boldsymbol{H}\varphi_j)(a) = (\boldsymbol{S}\varphi_j)(a) - (\boldsymbol{SA}\varphi_j)(a). \tag{3.12}$$

Now let $j \to \infty$. The first term is

$$(\boldsymbol{H}\varphi_j)(a) = \frac{1}{2\pi \mathrm{i}} \int_{\xi \in \partial\Omega} \frac{\dot{\gamma}(\xi)}{\xi - a} \varphi_j(\xi) \mathrm{d}\sigma_\xi$$

and has limit

$$\frac{1}{2\pi \mathrm{i}} \frac{\dot{\gamma}(z)}{z - a}. \tag{3.13}$$

Similarly,

$$(\boldsymbol{S}\varphi_j)(a) = \int_{\xi \in \partial\Omega} S(a, \xi) \varphi_j(\xi) \mathrm{d}\sigma_\xi$$

tends to $S(a, z)$ in view of the continuity of $S(a, \xi)$. The third term, after a change of the order of integration (the integrand is continuous in both variables z and w), is

$$(SA\varphi_j)(a) = \int_{\xi\in\partial\Omega}\left[\int_{w\in\partial\Omega} A(w, \xi)S(a, w)\mathrm{d}\sigma_w\right]\varphi_j(\xi)\mathrm{d}\sigma_\xi$$

and tends to

$$\int_{w\in\partial\Omega} A(w, z)S(a, w)\mathrm{d}\sigma_w.$$

Collecting results, (3.12) yields in the limit

$$\frac{1}{2\pi\mathrm{i}}\frac{\dot{\gamma}(z)}{z-a} = S(a, z) - \int_{w\in\partial\Omega} A(w, z)S(a, w)\mathrm{d}\sigma_w, \quad z\in\partial\Omega. \tag{3.14}$$

Conjugating (3.14) and using $S(a, z) = \bar{S}(z, a)$ and $A(w, z) = -\bar{A}(z, w)$, we obtain (2.8). Hence, $S(z, a)$ *is* a solution of the integral equation.

Uniqueness: The kernel $A(w, z)$ is continuous and skew-hermitian on $\partial\Omega\times\partial\Omega$. Hence, the associated integral operator is compact from $L^2(\partial\Omega)$ to $L^2(\partial\Omega)$ and has a purely imaginary spectrum. Consequently (2.8) has a unique solution even in the class $L^2(\partial\Omega)$. The proof of Theorem 3 is finished.

3.5. Miscellaneous proofs and comments

The fact that $\partial\Omega$ is of class C^2 and not necessarily C^∞ introduces no complications: we know a priori that the Riemann mapping function R is in $C^1(\bar{\Omega})$. See, e.g., [11]. The usual argument leading to (2.1) is then valid. See, e.g., [2] and [8]. In fact,

$$S_\Omega(w, z) = \sqrt{R'(w)}\,S_D(R(w), R(z))\overline{\sqrt{R'(z)}}, \quad w,z\in\Omega, \tag{3.15}$$

where S_Ω and S_D are the Szegö kernels of Ω and of the unit disk D respectively. Since (see [2] and [8]) $S_D(\xi, \eta) = 1/2\pi(1-\xi\bar{\eta})$ and R' is continuous on $\bar{\Omega}$ it follows that $S(w, z)$ has a continuous extension to $B = \bar{\Omega}\times\bar{\Omega} - \{(z, z);\ z\in\partial\Omega\}$ inheriting the hermitian property $S(w, z) = \bar{S}(z, w)$ (which is usually first justified only for $w\in\Omega$ and $z\in\Omega$). This proves the first part of Theorem 1 as well as the statements in Section 3.1.

The classical formula (2.1) follows from (3.15). Formula (2.2) follows in turn from (2.1): In fact, for $z\in\partial\Omega$, $\dot{\gamma}(z)R'(z)$ is tangent to the unit circle at the point $R(z)$, and the outer normal, which is precisely $R(z)$, must then be

$$(1/\mathrm{i})\dot{\gamma}(z)(R'(z)/|R'(z)|).$$

The proof of Theorem 2 for $\partial\Omega\in C^\infty$ is in [8]. The analysis in that proof can be routinely extended to the C^2 or C^k case ($k\geqslant 2$) with loss of two derivatives.

4. Implementation

We solve the second kind integral equation (2.9) by Nyström's method [1]. Choosing n equidistant collocation points $t_i = (i/n)\beta$, we use the trapezoidal rule to discretize (2.9):

$$\varphi(t_i) + \frac{\beta}{n}\sum_{j=1}^{n} k(t_i, t_j)\varphi(t_j) = \psi(t_i). \tag{4.1}$$

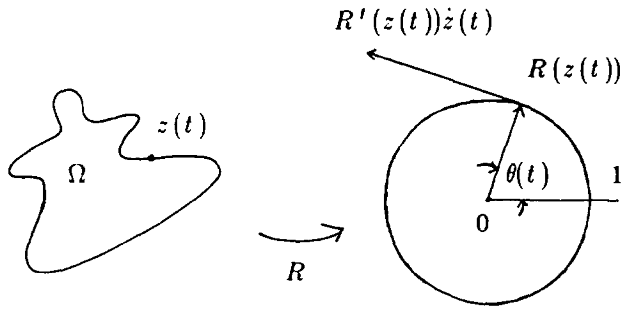

Fig. 2. Boundary correspondence function θ.

Note that the functions φ, ψ and k are periodic (in both arguments) with period β. Therefore, the trapezoidal rule can be expected to yield good results, especially under additional smoothness assumptions on the kernel k. (4.1) gives rise to a system of n linear equations in n complex unknowns

$$(I+B)x=y, \tag{4.2}$$

where the skew-hermitian matrix $B(=-B^{-\mathrm{T}})$ is defined by $B_{ij}=(\beta/n)k(t_i,\ t_j)$. The complex system can be rewritten as a $2n$ by $2n$ real system; however, approximately twice as many real multiplications are necessary to solve the larger real system as the complex system. The eigenvalues of $I+B$ are on the line $\mathrm{Re}\lambda=1$; in particular, (4.2) has a unique solution. According to [4], this system can be solved by Gaussian elimination without pivoting.

Once the discretized solution $x_i=\varphi(t_i)$ is known at the collocation points, (2.9) provides us with a natural interpolation formula for φ (see e.g. [1]):

$$\varphi(t)=\psi(t)-\frac{\beta}{n}\sum_{j=1}^{n}k(t,\ t_j)x_j. \tag{4.3}$$

Although rather expensive to evaluate, (4.3) is indeed a very good interpolation formula, as can be seen in Section 5.

The boundary correspondence function $\theta(t)$ (see Fig. 2) is defined by

$$R(z(t))=\mathrm{e}^{\mathrm{i}\theta(t)}, \tag{4.4}$$

where again R denotes the Riemann mapping function. Differentiating (4.4) yields

$$R'(z(t))\dot{z}(t)=\mathrm{i}\ \mathrm{e}^{\mathrm{i}\theta(t)}\dot{\theta}(t). \tag{4.5}$$

Using (2.1), (2.10), and (4.5) we can compute $\theta(t)$ (without integration) by the following formula:

$$\theta(t)=\arg\left[-\mathrm{i}\phi^2(t)\dot{z}(t)\right]. \tag{4.6}$$

If, in addition, one is interested in the derivative $\dot{\theta}(t)$, taking absolute values in (4.5) yields

$$\dot{\theta}(t)=(2\pi/S(a,\ a))\,|\phi^2(t)|. \tag{4.7a}$$

Since $\int_0^\beta\dot{\theta}(t)\mathrm{d}t=2\pi$, integrating (4.7a) with respect to t gives

$$S(a,\ a)=\int_0^\beta|\phi^2(t)|\mathrm{d}t. \tag{4.7b}$$

(4.7b) can also be obtained by using the reproducing property of the Szegö kernel:

$$\begin{aligned} S(a, a) &= \int_{\partial\Omega} S(a, z)S(z, a)\mathrm{d}\sigma_z \\ &= \int_0^\beta \overline{S(z(t), a)}S(z(t), a)\,|\dot{z}(t)|\,\mathrm{d}t \\ &= \int_0^\beta |\varphi(t)|^2\mathrm{d}t. \end{aligned}$$

There is a second way to compute $S(a, a)$, namely by the Cauchy formula:

$$\begin{aligned} S(a, a) &= \int_{\partial\Omega} H(a, z)S(z, a)\mathrm{d}\sigma_z \\ &= \int_0^\beta H(a, z(t))S(z(t), a)\,|\dot{z}(t)|\,\mathrm{d}t \\ &= \int_0^\beta \overline{\psi}(t)\varphi(t)\mathrm{d}t. \end{aligned} \tag{4.7c}$$

(4.7c) can also be obtained from (4.7b) by taking inner products with φ in (2.9) and noting that k is skew-hermitian. If (4.7b) and (4.7c) are evaluated numerically, then the discrepancy between these two numbers (as well as the size of the imaginary part in (4.7c)) generally indicates the quality of the approximation.

If one wants to determine the inverse boundary correspondence function $t(\theta)$, the knowledge of $\dot{\theta}$ might be of great advantage. A reasonable procedure to compute $t(\theta)$ is as follows:

- Solve $\theta(t) = \theta_j$ for equidistant $\theta_j = 2\pi j/m$ using Newton's method ($\dot{\theta}$ is known!).
- Approximate $t(\theta)$ by a trigonometric polynomial interpolating at θ_j. The Fast Fourier transform can be used to compute the coefficients [3,6].

Remark 1. The stability of the method is tied to the question of the stability of the linear system $(I + B)x = y$. Let $\|\ \|$ denote the operator norm (with respect to the Euclidean norm) of a matrix acting on $\mathbb{C}^n$. This norm is always bounded by the Frobenius or Hilbert–Schmidt norm $\|\ \|_F$ defined by

$$\|B\|_F := \left[\sum_{i,j} |b_{ij}|^2\right]^{1/2}.$$

Since all eigenvalues of $I + B$ have real part 1, $\|[I + B]^{-1}\| \leqslant 1$. Therefore the condition number of $I + B$ is bounded by

$$\operatorname{cond}(I + B) \leqslant \|I + B\| \leqslant 1 + \|B\| \leqslant 1 + \|B\|_F.$$

In most of our examples (but not in all), $\|B\|_F$ was less than 1, making (4.2) very well conditioned.

Remark 2. We thank a referee for his observation that, the system of equations being complex, the computing time is multiplied by a factor of 4 with respect to the usual real integral equations in conformal mapping. Hence the interest in investigating whether a coarser grid yields the same

Table 1(a)
Error norm $\| \theta_n - \theta \|_\infty$ for Example 1. Ellipse with axis ratio q

n	$q = 2:1$	$3:1$	$5:1$
4	0.17	1.95	2.5
8	0.016	0.25	3.6
16	0.00019	0.013	1.7
32	0.00000035	0.000052	0.04

Table 1(b)
Error norm $\| \theta_n - \theta \|_\infty$ for Example 1. Ellipse with axis ratio q. With parameter transformation.

n	$q = 2:1$	$3:1$	$5:1$
	$\omega = -0.55$	-0.7	-0.85
8	0.00024	0.0042	1.1
16	0.000048	0.00075	0.028
32	0.00000048	0.000006	0.0001

accuracy. However, it should be noted that this particular method gives more information, namely the complex derivative of the Riemann mapping on the boundary. Therefore, one gets directly both $\dot{\theta}$ and θ, whereas the usual equations give only one of these functions.

5. Examples and numerical results

We apply the algorithm to four test regions, subject to the normalization $R(0) = 0,\ R'(0) > 0$.

Example 1. Ω is the interior of an ellipse with axes $1 + \epsilon$ and $1 - \epsilon$, $0 \leqslant \epsilon < 1$. $\partial\Omega$: $z(t) = e^{it} + \epsilon e^{-it}$, $0 \leqslant t \leqslant 2\pi$. The exact boundary correspondence is given by (see e.g. [7])

$$\theta(t) = t + 2 \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{\epsilon^m}{1 + \epsilon^{2m}} \sin(2mt).$$

We tried different parametrizations using the parameter transformation $s(t) = t + \frac{1}{2}\omega \sin(2t)$, $|\omega| < 1$. A choice of ω which makes the collocation points more equally distributed on the boundary curve (we use equidistant points in the parameter interval) yields better results. The largest errors occur near the points $t = 0$ and $t = \pi$. We computed the boundary correspondence for ellipses with axis ratio $q = 2:1$, $3:1$, and $5:1$ ($\epsilon = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$). The error norms for the original parametrization are listed in Table 1(a), for the new parametrization in Table 1(b).

Example 2. Inverted Ellipse [5]. Ω is the region obtained by reflecting the exterior of an ellipse with axes 1 and $1/p$ $(0 < p \leqslant 1)$ in the unit circle: $\partial\Omega$: $z(t) = \sqrt{(1 - (1 - p^2)\cos^2 t}\ e^{it}$. The exact boundary correspondence is given by $\tan t = p \tan \theta(t)$. See Table 2 for the results.

Table 2
Error norm $\|\theta_n - \theta\|_\infty$ for Example 2, 'Inverted Ellipse'.

n	$p=$ 0.8	0.5	0.2
4	0.0043	0.13	1.1
8	0.000087	0.023	0.24
16	0.00000024	0.00068	0.14
32	0.00000047	0.000005	0.028

Example 3. Apple (epitrochoid). Ω is the interior of the curve

$$\partial\Omega:\ z(t) = e^{it} + \tfrac{1}{2}\alpha\, e^{2it}, \qquad 0 \leqslant t \leqslant 2\pi,\ 0 \leqslant \alpha < 1.$$

For $\alpha = 1$ the boundary curve is the cardioid with a cusp at $-\frac{1}{2}$. The boundary correspondence is given by $\theta(t) = t$. In this example the largest errors occur near $t = \pi$; the method anticipates the upcoming singularity as α approaches the critical value 1. Table 3 shows the results.

Example 4. Eccentric Circle [5]. Ω is the circle $|z - c| < r$, $0 < c < r$; the boundary curve in polar coordinates is $\partial\Omega$: $z(t) = \rho(t)\, e^{it}(0 \leqslant t \leqslant 2\pi)$, where

$$\rho(t) = c \cos t + \sqrt{r^2 - c^2 \sin^2 t}\,,$$

with exact boundary correpondence

$$\tan \theta(t) = \sin t \Big/ \left(\frac{c}{r^2 - c^2} \rho(t) + \cos t \right).$$

Of course, this looks like the ideal region for the proposed method, since $A(w, z) \equiv 0$. The only thing that has to be done is to evaluate the right hand side of (2.9), which is essentially the Cauchy kernel. We show the error norms for the circles with $c = 1$ and $r = 2$. 1.01, 1.001, respectively, in Table 4. As the pre-image of the origin moves closer to the boundary, the approximations get worse.

We list the sup norm error $\|\theta_n - \theta\|_\infty$, where θ_n is the approximation obtained with n collocation points, and θ is the exact boundary correspondence function. The sup norm has been computed by evaluating θ at 36 equally spaced points in the parameter interval, most of which are *not* collocation points. The L^2-error $\|\theta_n - \theta\|_2$ is bounded by $\sqrt{\beta}\, \|\theta_n - \theta\|_\infty$.

The program runs on an Apple II computer and is written in Pascal. All operations are single precision (23 bits, or approximately 7 decimal places). The number of operations (i.e. real multiplications) involved is $\frac{4}{3}n^3 + \kappa n^2$. The first term comes from the solution of the complex

Table 3
Error norm $\|\theta_n - \theta\|_\infty$ for Example 3, 'Apple'

n	$\alpha=$ 0.3	0.6	0.9
4	0.00014	0.01	0.21
8	0.0000017	0.00084	0.074
16	0.0000012	0.0000041	0.019
48	0.00000048	0.00000048	0.000032

Table 4
Error norm $\|\theta_n - \theta\|_\infty$ for Example 4, 'Eccentric circle' $|z-1|<r$

$r=2$	1.01	1.001
0.0000004	0.00002	0.00021

system of linear equations; κ depends on the functions $z(t)$ and $\dot{z}(t)$, and is fairly large. See [17] for an efficient $O(n^2)$-implementation of this method, which allows one to handle up to 1000 collocation points within reasonable computing time.

Two interesting observations can be made: (1) A significant improvement is achieved in Example 1 by using a different parametrization. (2) The kernel A seems to carry enough information to yield good approximations even for small values of n.

Added in proof

(1) The only other reference we know of concerning the numerical implementation of the ideas in [8] is the 1980 Oberwolfach abstract [14]. This deals with a different method. We thank Professor W. Wendland for pointing out this reference.

(2) Fore a more efficient implementation of the linear algebra part of our method and for an analysis of convergence rates, see the forthcoming papers [17,15,16].

Acknowledgements

P. Henrici encouraged us all along; he also showed us another way to verify our integral equation independently of the idea that originates it (this will appear in [7]). J. Bunch and W. Gragg provided valuable references. L.N. Trefethen and the referees' comments helped improve the presentation of this paper. To them all, our warm thanks.

References

[1] K.E. Atkinson, *Survey of Numerical Methods for the Solution of Fredholm Integral Equations of the Second Kind* (SIAM, Philadelphia, 1976).
[2] S. Bergman, The kernel function and conformal mapping, *AMS Math. Surveys* **5** (1970).
[3] J.-P. Berrut, Thesis, ETH Zürich, 1984; and, A Fredholm integral equation of the second kind for conformal mapping, *J. Comput. Appl. Math.* **14** (1986) (this volume) 99–110.
[4] A. Buckley, A note on matrices $A = I + H$, H skew symmetric, *ZAMM* **54** (1974) 125–126.
[5] D. Gaier, Konstruktive Methoden der konformen Abbildung (Springer, Heidelberg, 1964).
[6] P. Henrici, Fast Fourier methods in computational complex analysis, *SIAM Rev.* **21** (4) (1979) 481–527.
[7] P. Henrici, *Applied and Computational Complex Analysis Pure and Appl. Math.* **3** (Wiley, New York, to appear).
[8] N. Kerzman and E.M. Stein, The Cauchy kernel, the Szegö kernel, and the Riemann mapping function, *Math. Ann.* **236** (1978) 85–93.
[9] V. Singh, An integral equation associated with the Szegö kernel function, *Proc. London. Math. Soc.* **10** (3) (1960) 376–394.

[10] E.M. Stein, *Boundary Behaviour of Holomorphic Functions in Several Complex Variables* (Princeton Univ. Press, 1972).
[11] S. Warschawski, On the Differentiability at the Boundary in Conformal Mapping, *Proc. AMS* **12** (1961) 614–620.
[12] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. **23** (American Mathematical Society, Providence, RI, 1939).
[13] H. Rademacher and Q. Toeplitz, *The Enjoyment of Mathematics* (Princeton University Press, 1957).
[14] M.R. Trummer, An efficient implementation of a conformal mapping method using the Szegö kernel, *Numer. Math.*, to appear.
[14] W. Wendland and W. Niethammer, Ein Iterationsverfahren zur Berechnung des Szegö-Kerns mit der integralgleichung von Kerzman und Stein. Mathematisches Forschungsinstitut Oberwolfach, Konstruktive Verfahren in der komplexen Analysis, Tagungsbericht 31/1980.
[15] J.-P. Berrut, Equivalence of the methods of Fourier and Nyström for solving Fredholm integral equations of the second kind (1985) submitted for publication.
[16] M.R. Trummer, Nyström's method versus Fourier type methods for the numerical solution of integral equations, Tech. rep. 84-23, Dept. of Comp. Science, Univ. of British Columbia, Vancouver, 1984.
[17] M.R. Trummer, An efficient implementation of a conformal mapping method based on the Szegö kernel, *SIAM J. Numer. Anal.* (1986) to appear.